A good series

Yue Kwok Choy

$$\sigma_1 = 1 + 2 + 3 + ... + n = \frac{1}{2} n(n+1)$$

$$\sigma_2 = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1) = \frac{1}{3} n(n+1)(n+2)$$

$$\sigma_3 = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + ... + n(n+1)(n+2) = \frac{1}{4} n(n+1)(n+2)(n+3)$$

:   :   :

$$\sigma_k = 1 \cdot 2 \cdots k + 2 \cdot 3 \cdots (k+1) + 3 \cdot 4 \cdots (k+2) + ... + n(n+1) \cdots (n+k-1) = \frac{1}{k+1} n(n+1) \cdots (n+k)$$

Proof

We can prove the σ_k formula by :

(1) Mathematical Induction – proof omitted.

(2) Difference method

The general term = $u_r = r(r+1) \cdots (r+k-1)$

Let $v_r = r(r+1) \cdots (r+k-1)(r+k)$. Then

$$\begin{aligned} v_r - v_{r-1} &= r(r+1) \cdots (r+k-1)(r+k) - (r-1)r(r+1) \cdots (r+k-1) \\ &= r(r+1) \cdots (r+k-1)[(r+k) - (r-1)] \\ &= r(r+1) \cdots (r+k-1)[k+1] = (k+1) u_r \end{aligned}$$

$$\therefore \quad u_r = \frac{1}{k+1}(v_r - v_{r-1})$$

$$\Rightarrow \sum_{r=1}^{n} u_r = \frac{1}{k+1} \sum_{r=1}^{n} (v_r - v_{r-1}) = \frac{1}{k+1}(v_n - v_0) = \frac{1}{k+1} n(n+1) \cdots (n+k)$$

Why the series σ_k formula is good ?

(1) $\sigma_1, \sigma_2, \ldots, \sigma_k$ formulas show certain pattern and are easy to remember .

(2) The formulas can be used to find some other finite series .

Example 1 Find **(1)** $s_2 = \sum_{r=1}^{n} r^2$ and **(2)** $s_3 = \sum_{r=1}^{n} r^3$

(1) $s_2 = \sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} [r(r+1) - r] = \sum_{r=1}^{n} r(r+1) - \sum_{r=1}^{n} r$

$= \sigma_2 - \sigma_1 = \frac{1}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(2n+1)$

(2) $r(r+1)(r+2) = r(r^2+3r+2) = r[r^2 + 3(r+1) - 1] = r^3 + 3r(r+1) - r$

$\Rightarrow \sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r(r+1) - \sum_{r=1}^{n} r \qquad \Rightarrow \sigma_3 = s_3 + 3\sigma_2 - \sigma_1$

$\Rightarrow \frac{1}{4}n(n+1)(n+2)(n+3) = s_3 + 3\left[\frac{1}{3}n(n+1)(n+2)\right] - \left[\frac{1}{2}n(n+1)\right]$

$\therefore \quad s_3 = \left[\frac{1}{2}n(n+1)\right]^2$

Example 2 Find the sum of the first n terms : $2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \ldots$

We wish to find $S = \sum_{r=1}^{n} (r+1)(2r+3)$.

Consider (r + 1) (2r + 3) ≡ A r(r + 1) + B r + C

We can use compare coefficient method, or otherwise, and find that : A = 2 , B = 3 , C = 3

$\therefore$ (r + 1) (2r + 3) ≡ 2 r(r + 1) + 3 r + 3

$\therefore \quad S = \sum_{r=1}^{n} (r+1)(2r+3) = 2\sum_{r=1}^{n} r(r+1) + 3\sum_{r=1}^{n} r + 3\sum_{r=1}^{n} 1 = 2 \times \frac{1}{3}n(n+1)(n+2) + 3 \times \frac{1}{2}n(n+1) + 3n$

$= \frac{1}{6}n[4n^2 + 21n + 35]$

Exercise

Show that : $1^2 + 3^2 + \ldots + (2n+1)^2 = \frac{1}{2}(n+1)(2n+1)(2n+3)$